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A Playful Geometry Workshop: Creating 3D Polyhedral Structures from Innovative 2D Self-assembling Paper Folding Units

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Abstract

Combining artistic creativity with highly simplified complex geometry, this workshop allows participants, teachers, parents, and children from age 7 and above, to experience first hand the folding movement of two-dimensional nets into three-dimensional polyhedral structures. Participants will become acquainted with the folding and unfolding processes of a broad range of complex polyhedral structures, in a simple, easy way. The building unit comprises eight equilateral triangles, one of 11 known nets of the octahedron, and its underlying mechanism allows not only for the formation of many known polyhedral (deltahedral) structures but also that of many unknown deltahedral formations. Using this model, complex polyhedral structures may be built quickly and easily in a matter of seconds devoid of the need of glue or scissors, angle calculations, or mathematical formulae. The result is an accessible world of polyhedrons available as a new self-assembling geometric model and a new folding game governed by a simple pedagogical technique by which both simple and complex multidimensional polyhedral structures may be formulated.

Polyhedral structures play a significant role in our lives, of which we are typically unaware, whether used for the amusement of the two year old toddler playing with her wooden cubes or studied by the educated mathematician [16]. We live in Platonic cubes, we are awestruck as we look at the Egyptian pyramids, and the viruses we catch are, in fact, living icosahedrons. Physical atoms from which the world is comprised are drawn to polyhedral structures. Thus, polyhedral structures bring together varied fields of knowledge, such as art, architecture, molecular biology, and mathematics to name a few. How often, though, have young student or adults explored this rich world of polyhedral structures?

A brief historical view of polyhedral structures and geometric nets

Polyhedra have a long history as a source of fascination in geometry as well as other scientific fields. Plato (ca. 343 BC) was the first to expose the five elementary convex structures (cube, tetrahedron, octahedron, icosahedron, and dodecahedron), claiming that these structures are no less than the key for everything that exists in our world. About a century later Archimedes (ca. 212 BC) explored integrations of the Platonic solids [5], discovering thirteen new semi-regular convex polyhedra, known today by the name of Archimedean solids [1]. More recently, Leonardo Da Vinci (1452-1519) had studied methods by which three-dimensional polyhedral structures could be drawn on two-dimensional layers of paper, resulting in his illustrations for Luca Pacioli's book *The Divine Proportion* which was published in 1509 [3]. The astronomer Johannes Kepler (1571-1630) explored the relationship between polyhedral structures and the solar system [8]. In 1966, mathematician Norman Johnson (1930---) published a list of 92 strictly convex polyhedra whose faces are all regular polygons, which are known by the name of Johnson solids [2]. Also in the 20th century, architect Buckminster Fuller (1895-1983) experimented with polyhedral structures in architecture resulting in the geodesic dome while mathematician and biochemical theorist Dorothy Maud Wrinch (1894-1976) suggested that protein ribbons fold into polyhedral shapes, thereby opening the door to what was then the new promising field of protein folding [8].

In geometry, a net is the unfolding of a single, simply connected, non-overlapping polygon comprising the faces of the polyhedron as they are attached at their edges [6]. Polyhedral nets date back to 1525, when the painter Albrecht Dürer found a way to represent polyhedrons in two-dimensional nets [9]. Dürer introduced the notion of polytope nets and published the nets of some Platonic and Archimedean polyhedra along with directions for their reconstruction [12]. In half a millennium that had elapsed since then, Dürer's nets have become the standard presentation method in describing polyhedra [13]. In 1971, mathematician Father Magnus J. Wenninger (1919---) made acceissbile the world of nets and their folding into polyhedral structures by publishing his book *Polyhedron Models* [16].

Despite these ancient roots, the field of nets is considered a relatively new field of research, bringing together the disciplines of spatial geometry and molecular physics and biology, where the concept of self-assembly is incredibly powerful [11]. Genetic engineers are pursuing a stimulating quest for the rules governing the self-assembling of protein ribbons into complex three-dimensional structures while nanotechnologists are seeking basic building blocks that can be folded into complex structures [7]. The ability researchers have achieved in building basic geometric structures such as polyhedral as well as in self-assembly laboratory experiments using molecules such as those which comprise our DNA, is striking [14].

All known polyhedral structures unfold into several different nets. For example, the tetrahedron unfolds into two nets, the cube and the octahedron unfold into eleven different nets each, and the dodecahedron and icosahedron can each be unfolded into 43,380 distinct nets [11, 15]. An important question thus emerges regarding the criteria we can employ that determine how we search efficiently for the optimal net among a vast multitude of available nets [11]. Lacking the availability of simple pedagogical tools, it has thus far been near impossible to explore this field effectively [10]. The model presented in this workshop unifies many polyhedral structures of the deltahedal (comprising equilateral triangles as faces) by unfolding them into a single net, one of the known nets of the octahedronal net, which serves as the building unit of all these structures. Moreover, the model presented herein allows for a simple and joyful exploration of this rich world of polyhedral structures underlying our very existence and the world at large. It allows both children and adults to experience first-hand these important polyhedral structures which we were never taught about in school by building them with our hands without any accessories. This is done on the basis of a magnetic attraction/rejection pattern.

The spiraling octahedral building unit of the model

The model presented herein has been transformed into a paper-based octahedral building unit comprising eight equilateral triangles and seven folding hinges; its magnetic attraction/rejection pattern has been translated to paper slits and slots (Figure 1). This net has proven to be the common denominator of many polyhedrons and other complex structures and has shown great flexibility when the single building unit is multiplied (see below). Among others, the unified net suggested herein, in its diverse tessellation variations, folds into three of the five Platonic solids, some of the Johnson solids, as well as many unnamed structures.

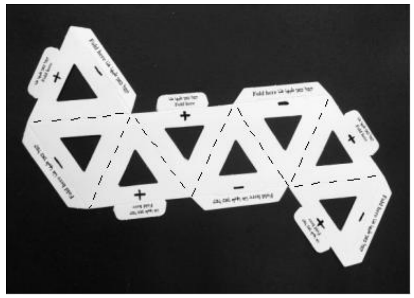


Figure 1: *The paper spiraling octahedral building unit and its magnetic attraction/rejection pattern translated to paper slits and slots (+ and – signs) and seven folding hinges (marked by a dashed line)*

The magnetic attraction/rejection pattern of the spiraling octahedral building unit

An indispensable characteristic of the building unit is its unique attraction/rejection pattern as manifested around its edges (Figure 1). The building unit functions as an autonomous unit that can interrelate with other building units or parts thereof, in numerous manners all of which are governed by this attraction/rejection pattern. This is true whether building units are tessellated in two dimensional spaces or whether they are folded in multidimensional space. Due to their attraction/rejection pattern, building units may be conjoined in whole or partial segments (Figure 2), and partial segments may be symmetrically or asymmetrically truncated (Figure 2a).

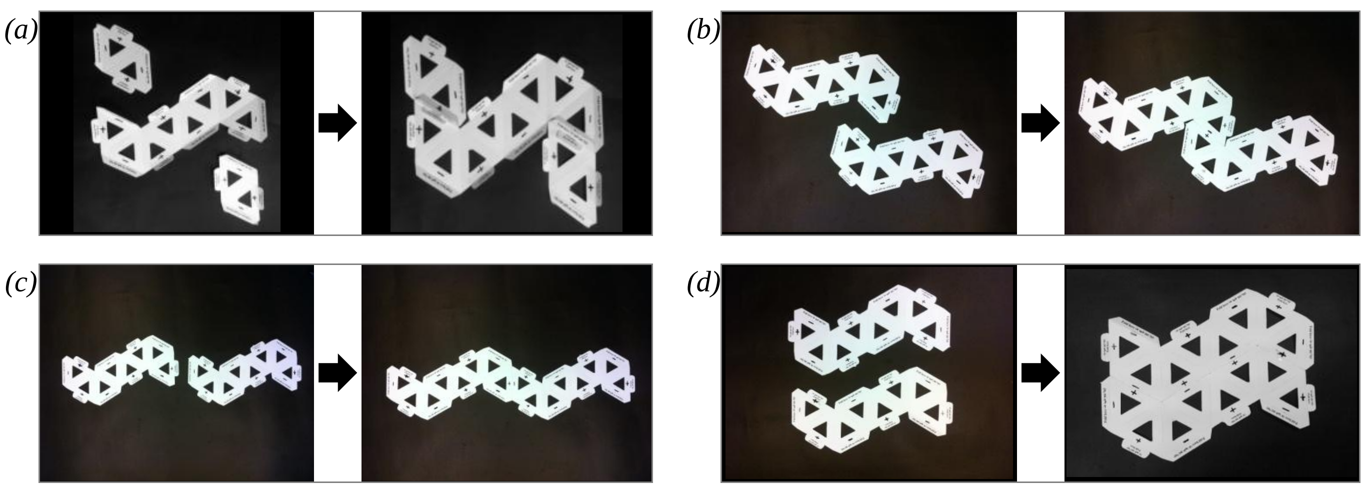


Figure 2: *Expanding the building unit: (a) segments appended to the building unit may be connected symmetrically as in the figure or asymmetrically by appending only one of the two shown segments;(b) a chained helix-like continuum of building units; (c) a vertical continuum of building units; (d) a parallel continuum of building units*.

The building unit may be connected with other units to form unit continuums in three different patterns. The chained helix-like connection (Figure 2b) links building units in a chained sequence, where the tail of one building unit integrates into the head of the next building unit, forming a unique spiraling chain of amalgamated units. The vertical connection (Figure 2c) links building units parallel to each other so that they form a rather narrow but elongated strip of amalgamated units. Finally, the parallel connection (Figure 2d) links building units next to each other so that they form a rather wide strip of amalgamated units.

The folding principle

As mentioned and illustrated, the building unit comprises eight equilateral triangles and seven folding hinges (Figure 1). Given its different tessellations and versions of extension (Figure 2), the building unit(s) may be folded in four different ways (Figure 3). Two of these (Figure 3c,d) form the basis for a family of three-dimensional structures that is quite different than the polyhedral structures discussed herein. However, these as well as additional types of tessellations and folding patterns are beyond the scope of the current paper. The spiraling folding governed by the attraction mode of the two poles of the chained helix-like ribbon (Figure 3b) is the method that will be practiced in the suggested workshop.

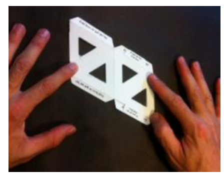
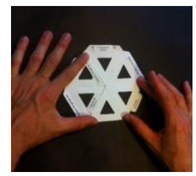
(a)  *(b)*  *(c)*  *(d)* 

Figure 3: *Folding patterns of building unit(s): (a) tetrahedral folding; (b) spiraling folding governed by the attraction mode of the two poles of the chained helix-like ribbon; (c) folding into a parallelogram building unit; (d) folding into a hexagon building unit*

The workshop

In this workshop participants will fold polyhedral structures, simply and quickly, to form an extensive series of paper polyhedral structures. The activity starts with a paper cut model of the basic building unit comprising eight equilateral triangles and various segments of the basic building unit (Figure 4).

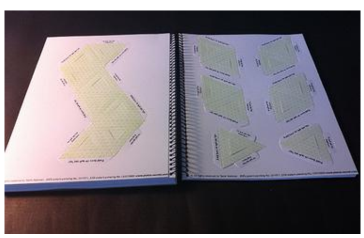
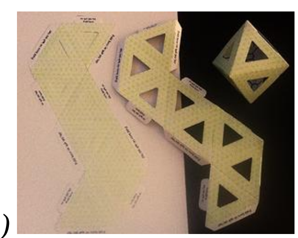
(a)  *(b)* 

Figure 4: *Paper cut model of the basic building unit and segments thereof*

By connecting and folding basic building units and segments thereof, participants experience and explore first hand the folding of two-dimensional nets into complex three-dimensional structures using a simple new technique dispensing of the need for glue or scissors. No previous knowledge is required of the folded polyhedral structures, which reveal themselves through the folding process. The model and its paper cut application may thus serve has a new pedagogical tool for math teachers as they endeavor to teaching spatial geometry [10]. Thanks to the model's specific attraction/rejection pattern such investigations are made possible overcoming the necessity of complex mathematical formulae or angle computation. The model presented herein thus offers a simple building unit which facilitates further experiential learning of the formation of polyhedral structures from two-dimensional nets.

This workshop focuses on two-dimensional net emerging from the chained helix-like connection of basic building units (Figures 2b, 3b). Chains of this type fold into a geometric series of polyhedral structures starting with the simplest tetrahedron and building towards rather complex structures such as the hexagonal anti-prism (Figure 5). The governing principle in this folding process is that any ribbon derived from the building unit(s) folds into a polyhedral structure as long as it contains an even number of triangles greater than four. Figure 5 illustrates the folding process of several chains that will feature in the workshop, each comprising whole building units while Figure 6 illustrates the folding process of chains mixing both whole and segmented building units. The folding principles of all are identical, involving the same five steps: (i) preparing the adequate number of basic paper building units; (ii) if necessary, attaching basic paper building units in a chained helix-like connection; (iii) simultaneously folding the two poles of the amalgamated chain in a counter-clock spiraling movement to attach them together; (iv) closing the ringed chain to form the final form; (v) enjoying the final polyhedral structure that emerged through this unified folding process.

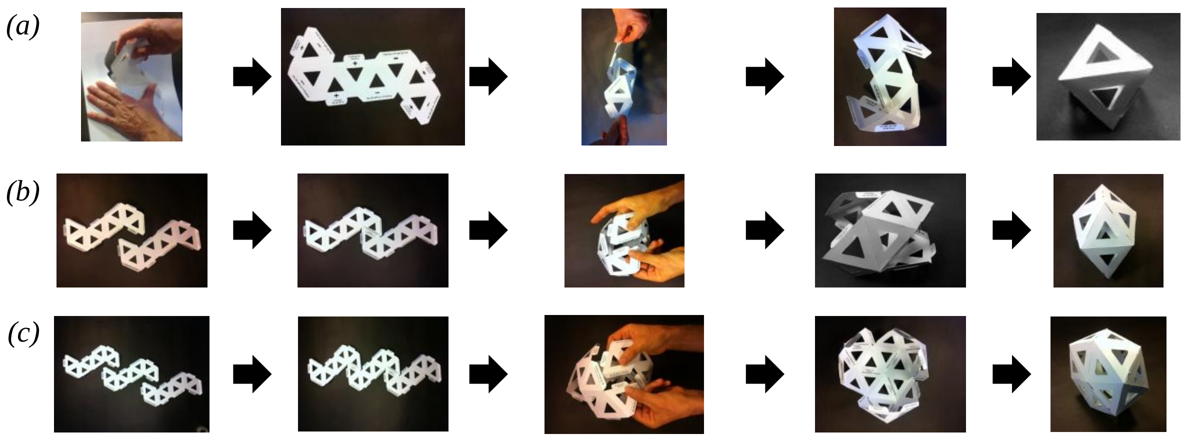


Figure 5: *Workshop activity – folding various chained helix-like ribbons of whole basic building unit(s): (a) a single building unit folds into the octahedron; (b) two building units fold into the Gyroelongated square dipyramid (Johnson's polyhedra 17); (c) three building units fold into the hexagonal anti-prism*

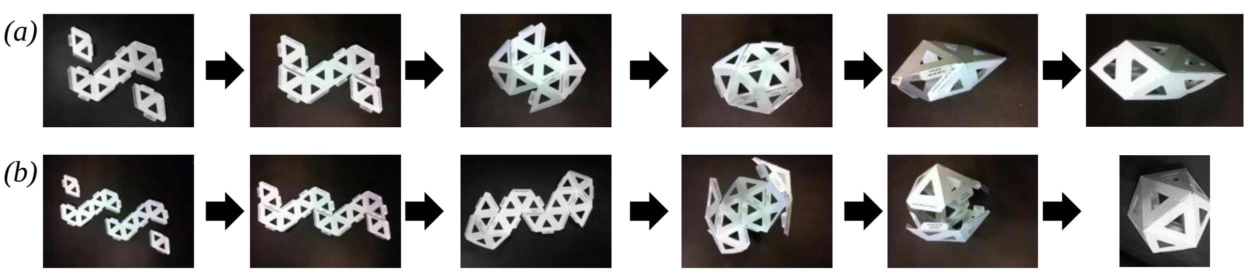


Figure 6: *Workshop activity – folding various chained helix-like ribbons of whole and segmented basic building unit(s): (a) a single building unit with two symmetrically appended units of two triangles each folds into the rombohedron; (b) two building units with two symmetrically appended units of two triangles each fold into the icosahedron*

Other polyhedral structures folded from vertical and a parallel continua (Figure 2c,d) of whole and segmented octahedral building units are shown in Figure 7. Some of these are known and named (Figure 7a-f) while others have not yet been named (Figure 7g-i).

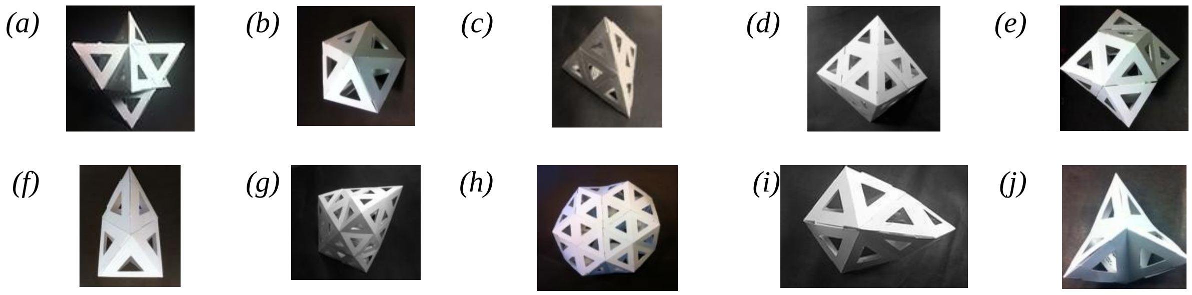


Figure 7: *Polyhedral structures folded from vertical and parallel continua of whole and segmented building units: (a) Stella octangulla; (b) decahedron; (c) large tetrahedron; (d) large octahedron; (e) pyramid star; (f) snub disphenoid – Johnson's polyhedra 84; (g-i) some unspecified structures*

During these 1.5 hours of a workshop, participants may choose to fold a multitude of polyhedral structures, from the simple tetrahedron to the complex icosahedron. Participants may take their creations home with them, decorate them, and turn them into a mobile that has the power to inspire imagination and creativity in their every day life (Figure 8).

Figure 8: *Imagination and creativity inspiring polyhedron mobile constructed from folded spiraling octahedral paper building units (tetrahedron, triangular bipyramid, octahedron, decahedron, rombohedron, Gyroelongated square dipyramid, icosahedron, hexagonal anti prism)*

Summary

It is a well-known problem that students find the visualization of three-dimensional shapes challenging, which leads to difficulties when teaching the names and characteristics of polyhedra [10]. The model presented herein unifies many polyhedral structures by providing for all an underlying single pattern in the form of the spiraling octahedral paper building unit described throughout this paper. Governed by a repetitive pattern of attraction/rejection characterizing its edges, the model allows for quick and intuitive folding process saving students from frequent frustrations resulting from accidentally cutting off tabs as well as from forgetting or misunderstanding which edges ought to be glued together [10]. Due to its easy and intuitive application, workshop participants will not only have the opportunity to construct many of the known polyhedral structures but they will also be able to seek and investigate new structures.

Another key advantage of this model is its high potential for mass production and thereby its potential to serve as a useful pedagogical tool by which to explain polyhedral structures that are significant in the fields of geometry, molecular biology, and architecture, to name but a few. Simply and easily, pupils and students in nature and biology classes may learn and imitate the structure of viruses or explore manifestation's of Euler's polyhedron formula V-E+F=2. This formula simplifies all polyhedral structures by tying the number of faces, edges, and vertices where the number of vertices less the number of edges and faces always equals 2. The model presented herein extends this formula to the world of two-dimensional nets [4]. The model may thus increase students' motivation and joy as they explore the beauty of mathematics through this simple model. Bridges between art and science, students may develop their spatial awareness and visualization skills as learn to transition from two-dimensional nets to three-dimensional polyhedral structures. Reducing this myriad of polyhedral structures into a single unifying net governed by its magnetic attraction/rejection pattern renders the model an effective and simple vehicle that may be used in self-assembly fields, applicative in the worlds of nanotechnology and structure deployment [7, 11, 14].

This experiential workshop is not only educational and fun but also serves to develop participants' sense of achievement (Figure 9). "It is really surprising how much enlightenment will come following the construction of the models rather than preceding it, and once you begin making them you may find that your enthusiasm will grow", said Father Magnus J. Wenninger with respect to polyhedron models [16]. I find these words of his to be an accurate reflection of the experience of workshop participants as they fold and unfold a wide variety of polyhedral structures. The octahedral net discussed in this work has yet to be studied, to be fully understood in the context of science and its relation to theoretical mathematical concepts. Treading pristine grounds, applicable consequences of the model and workshop proposed herein are still vague. At a very practical level, however, the model presented herein offers a simple building unit that facilitates further experiential investigation of the formation of polyhedral structures from 2D nets. Thanks to its specific attraction/rejection pattern such investigations are possible without the need of complex mathematical formulae, angle calculations, or even glue thereby preserving student's curiosity and joy in playful exploration.

Figure 9: *Educational and fun experiential workshop that also serves to develop a sense of achievement; 4th grade children participating in a recent workshop*

About the workshop facilitator

Tamir Ashman, the workshop facilitator, is an independent student of geometry. For the past thirteen years he has investigated the stella octangula (see above, Figure 7a), a process which gave birth to the octahedral paper building unit and the paper game model which are at the heart of this workshop. A short video clip demonstrating the folding process of the octahedral building unit may be found here: http://www.youtube.com/watch?v=V4-7HwfsTD8.
For more information about the folding instructions, visit our website: http://www.platos-secret.com

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